Regularized semiparametric models on planar linear networks

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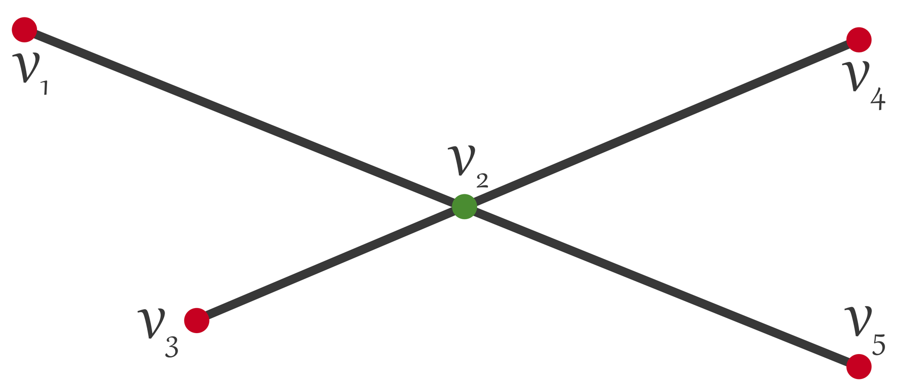
Functional Spaces on Planar Linear Netorks

A planar linear network $\mathcal{G} = (W, E)$ can be characterized by the set of vertices W and the set of edges E:

$$\begin{cases} W = \{v_1, \dots, v_\ell\} \\ E = \{e_1, \dots e_k\} \end{cases}$$

Moreover, we can split the vertices $W = W_I \cup W_B$, where W_I denotes the set in *interior* vertices, while W_B the set of *boundary* vertices.



To each edge incident to v_ℓ we assign a positive or negative sign according to whether the edge ends at v_ℓ or starts at v_ℓ:

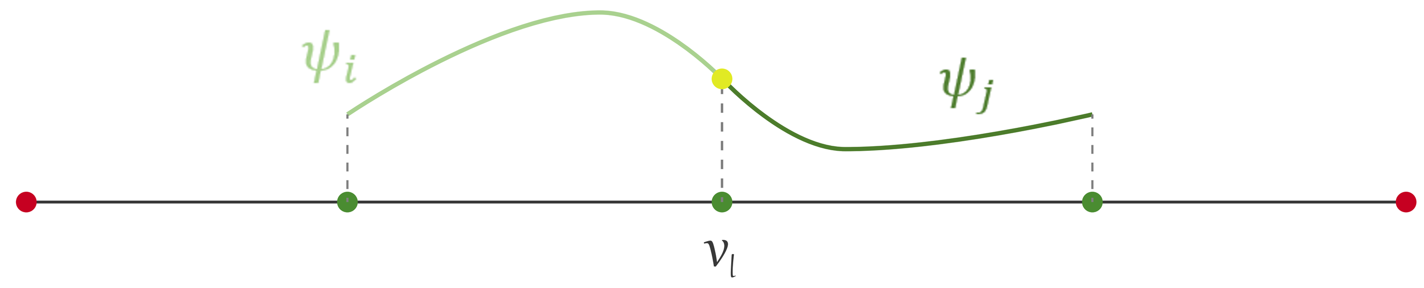
$\delta_{i\ell} = 1$, if e_i ends at v_ℓ; $\delta_{i\ell} = 1$, if e_i starts at v_ℓ.

We define the L^2 space over the network:

$$L^2(\mathcal{G}) := \{\phi : \mathcal{G} \to \mathbb{R} \text{ s.t. } \phi_i \in L^2(e_i)\ \forall e_i \in E\}$$

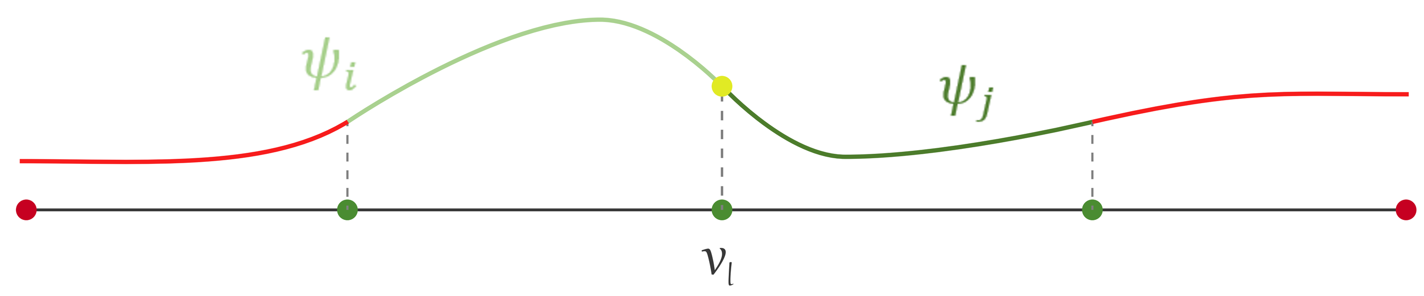
Similarly, imposing appropriate transmission conditions, we define the Sobolev space

$$H^2(\mathcal{G}) := \{\phi : \mathcal{G} \to \mathbb{R} \text{ s.t. } \phi_i \in H^2(e_i)\ \forall i \in I;\ \phi_i(v_\ell) = \phi_j(v_\ell)\ \forall i, j \in I_\ell, v_\ell \in W_I; \sum_{i \in I_\ell} \delta_{i\ell}\phi_i'(v_\ell) = 0,\ \forall v_\ell \in W_I\}$$



Finally, we define the space V with Neuman boundary conditions:

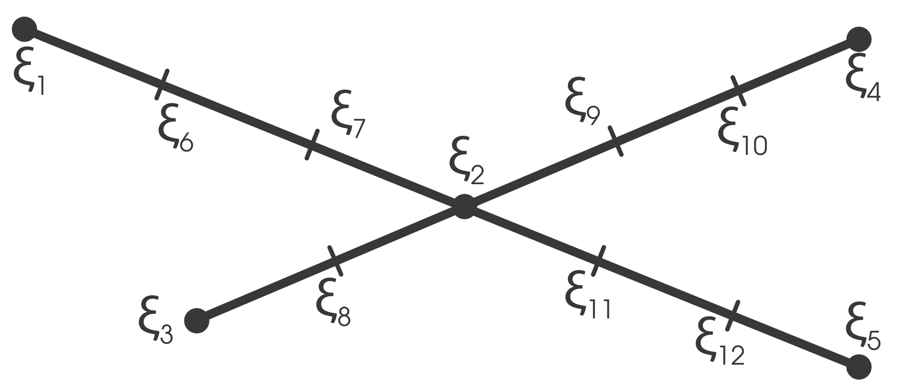
$$V := \{\phi : \mathcal{G} \to \mathbb{R} \text{ s.t. } \phi_i \in H^2(\mathcal{G})\ \forall i \in I;\ \phi_i'(v_\ell) = 0\ \forall i \in I,\ v_\ell \in W_B\}$$



Finite Elements

We define a refined version of the network $\mathcal{G}_\tau = (W_\tau, E_\tau)$

$$\begin{cases} W_\tau = \{\xi_1, \dots, \xi_{N_\tau}\} \\ E_\tau = \{e_1, \dots e_{K_\tau}\} \end{cases}$$

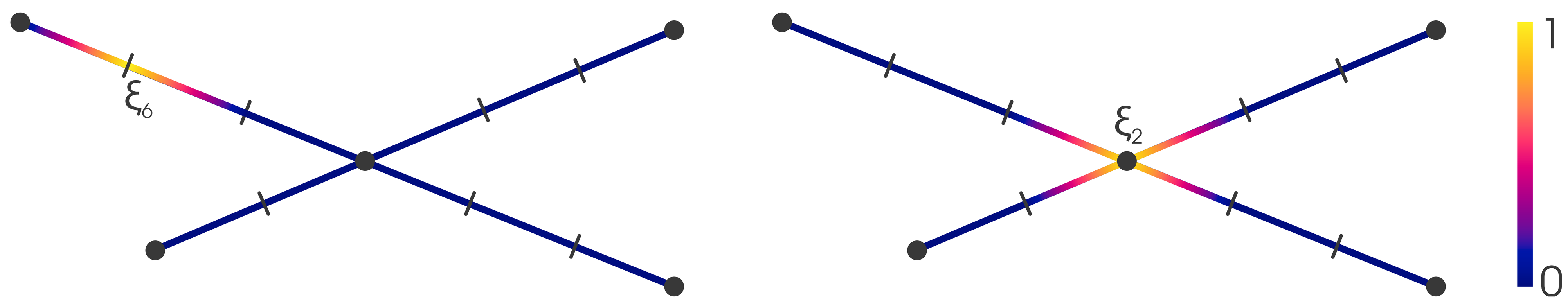


We define a set of N_τ basis functions $\psi_1, \dots, \psi_{N_\tau}$, each associated to a node ξ_i, such that

- ψ_i is linear over each edge $e_i \in E_\tau$,
- $\psi_i(\xi_j) = 1$ if $i = j$, and 0 otherwise.

We discretize the space V with V_τ

$$V_\tau = \{\psi_\tau \in C(\mathcal{G}) \text{ s.t. } \psi_\tau|_{e_\tau} \in \mathbb{P}^1\ \forall e_\tau \in E_\tau\}.$$



Density Estimation

The aim of the problem is to estimate a density function over a planar network. Let

- $f : \mathcal{G} \to \mathbb{R}$ be a density function
- $\{x_1, ..., x_n\}$ be n independent realizations from f

We propose to estimate $g = \log(f)$ by minimizing the penalized negative log-likelihood functional

$$L(g|x_1, ..., x_n) = -\frac{1}{n}\sum_{i=1}^n g(x_i) + \int_{\mathcal{G}} e^g + \lambda \int_{\mathcal{G}} (\Delta g)^2.$$

Well Posedness

Theorem 1 The functional $L(g)$ has a unique minimizer in V.

Lemma 1 The functional $J(g) = -\frac{1}{n}\sum_{i=1}^n g(X_i) + \int_{\mathcal{G}} \exp(g)$ is continuous and strictly convex in V.

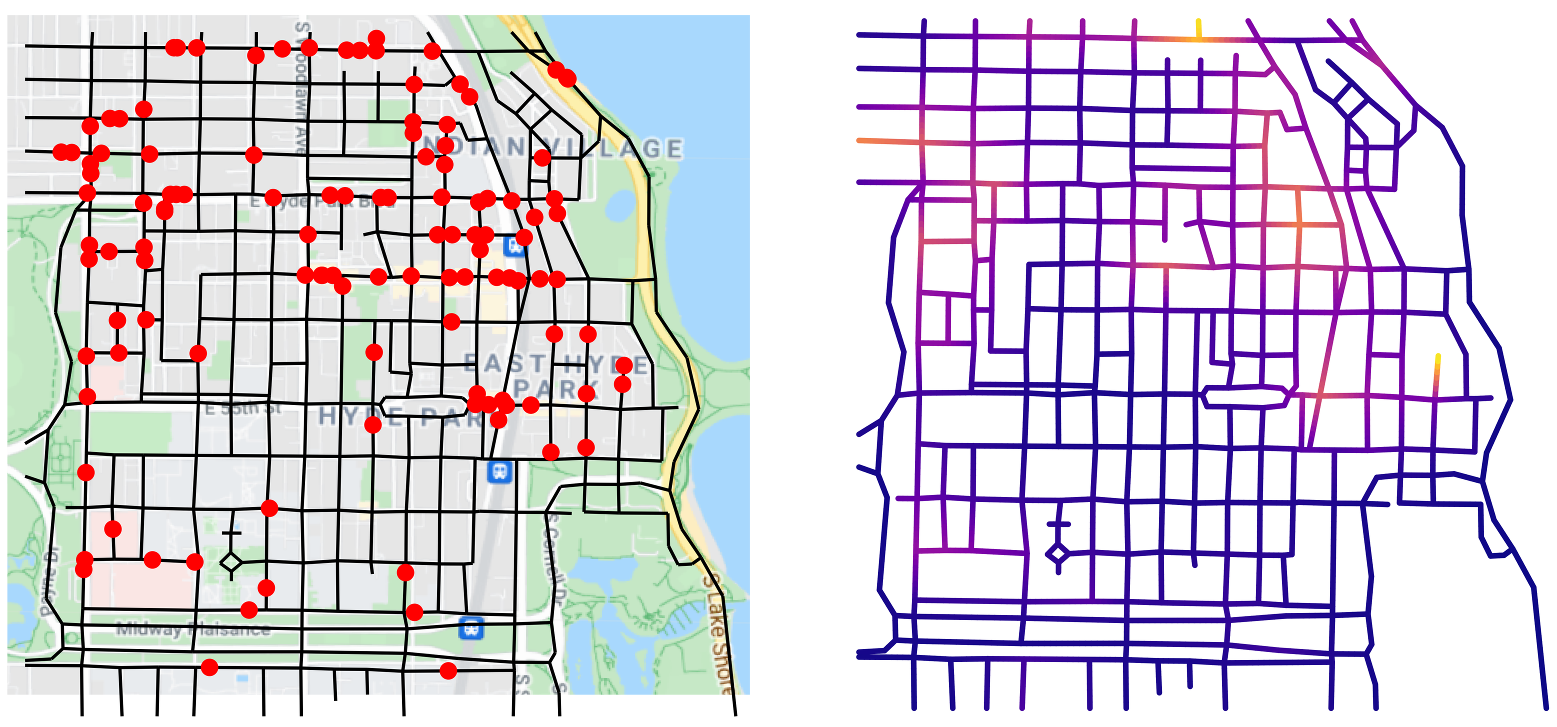
Lemma 2 Let $V_0 = \{g \in V : \Delta g = 0\}$ and V_Δ such that $V = V_0 \oplus V_\Delta$. V_0 is of finite dimension. Moreover $\|\Delta \cdot\|_{L^2}$ is a norm in the space V_Δ, equivalent to the H^2 norm.

Consistency

Assumption 1 The true log-density g_0 is bounded above and below, and is such that $\int_{\mathcal{G}} (\Delta g_0)^2 < \infty$.

Assumption 2 For g in a convex set B_0 around g_0 containing $\hat{g}$ and g_*, there exists a positive constant c such that $c\,\mathrm{Var}g_0 \leq \mathrm{Var}g$ uniformly with respect to g.

Theorem 2 Under the previous assumptions, as $\lambda \to 0$ and $n\lambda^{1/2} \to \infty$ the estimator $\hat{g}$ that minimizes $L(g)$ is consistent.



We apply the method to the Chicago crimes dataset from library `spatstat`. It records the nearest street address locations of crimes reported between 25 April 2002 and 8 May 2002 in the neighbourhood of the University of Chicago. On the left the observed data, while on the right the estimate obtained with the proposed method.

Spatial Regression

The aim is to estimate the parametric part $\boldsymbol{\beta}$ and nonparametric part $f : \mathcal{G} \to \mathbb{R}$ of a regression model where

- $\mathbf{p}_i \in \mathcal{G}$ for $i = 1, \dots, n$, are the locations of observation;
- $\mathbf{x}_i$ are observed covariates in $\mathbf{p}_i$;
- $y_i = \mathbf{x}_i^\top \boldsymbol{\beta} + f(\mathbf{p}_i) + \varepsilon_i$ models the observed data.

We propose to estimate $\boldsymbol{\beta}$ and f by minimizing the penalized sum-of-square-errors functional

$$J(f, \boldsymbol{\beta}) = \frac{1}{n}\sum_{i=1}^n \left(y_i - f(\mathbf{p}_i) - \mathbf{x}_i^\top \boldsymbol{\beta}\right)^2 + \lambda \int_{\mathcal{G}} (\Delta f)^2.$$

Denote with X the matrix of covariates, $Q = I - X(X^\top X)^{-1}X^\top$, and Ψ the matrix containing the evaluation of the basis function on the data location.

Well Posedness

Theorem 3 The pair $(\hat{\boldsymbol{\beta}}, \hat{f})$ that minimize $J(f, \boldsymbol{\beta})$ exists unique. Moreover:

- $\hat{\boldsymbol{\beta}} = (X^\top X)^{-1}X^\top(\mathbf{y} - \mathbf{f}_n)$
- f satisfies: $\mathbf{u}_n Q \mathbf{f}_n + \lambda \int_{\mathcal{G}} \Delta u \Delta f = \mathbf{u}_n Q \mathbf{y}, \quad \forall u \in V$

Consistency

Assumption 3 The matrices $A_n = n(\Psi^\top Q \Psi)^{-1}$ and $\Sigma_n = X^\top X/n$ exist. Moreover, their limits $\lim_n A_n$ and $\lim_n \Sigma_n$ exist.

Theorem 4 Let $n \to \infty$ and $\lambda = o(n^{-1/2})$. Under the previous assumption the discrete estimators $\hat{f}$ and $\hat{\boldsymbol{\beta}}$ are consistent and asymptotically Gaussian.

Changing the first term of the functional $J(f, \boldsymbol{\beta})$ the model can be extended in various direction, such as *generalized linear regression* or *quantile regression*.

References

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